BENHA UNIVERSITY

FACULTY OF ENGINEERING
AT SHOUBRA

PHYSICAL AND MATHEMATICAL DEPARTMENT

ENGINEERING MECHANICS

SOLVED PROBLEMS

FOR
PREPARATORY YEAR

BY

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Chapter I- Curvilinear Motion

1) A particle describes a curve (for which **s** and $\boldsymbol{\Psi}$ vanishes simultaneously) with uniform speed **V**. if the acceleration at point be $\left(\frac{c\,V^2}{(c^2+s^2)}\right)$, prove that the curve is a **catenary**

Solution

Since the particle move with uniform speed so

$V = \dot{s} = constant$ (1)

By differentiating with respect to t, we can get

$a_r = \ddot{s} = 0$ (2)

Since $a = \sqrt{(a_r)^2 + (a_n)^2} = \frac{c\,V^2}{(c^2+s^2)}$ (3)

From equation (2) and (3) get

$a = a_n = \frac{V^2}{\rho} = \frac{c\,V^2}{(c^2+s^2)}$ (4)

So $\rho = \frac{(c^2+s^2)}{c}$ (5)

Since $\rho = \frac{ds}{d\Psi}$ (6)

From equation (5) and (6) and separating variables can get

$\int \frac{ds}{(c^2+s^2)} = \int d\Psi$ (7)

By integrating both side the following form can get

$\frac{1}{c}\tan^{-1}\frac{s}{c} = \frac{\Psi}{c}$ (8)

So $s = c\tan\Psi$ the curve is a **catenary**

2) Prove that the acceleration of a point moving in a curve with uniform speed is **$\rho\dot{\Psi}^2$**.Such that **ρ** is the curvature and **Ψ** is the inclination angle.

If a point describes a curve $s = 4a\sin\Psi$, find its acceleration at any point **S**

<u>Solution</u>

Since the particle move with uniform speed so

$V = \dot{s} = constant$ (1)

By differentiating with respect to t, we can get

$a_r = \ddot{s} = 0$ (2)

Since $a = \sqrt{(a_r)^2 + (a_n)^2}$ (3)

From equation (2) and (3) get

$a = a_n = \frac{V^2}{\rho}$ (4)

Since $\dot{s} = \frac{ds}{dt} * \frac{d\Psi}{d\Psi} = \frac{d\Psi}{dt} * \frac{ds}{d\Psi} = \dot{\Psi}\frac{ds}{d\Psi}$ (5)

Since $\rho = \frac{ds}{d\Psi}$ (6)

So $\dot{s} = \rho\dot{\Psi}$ (7)

From equation (4) and (7) get

$a = \frac{V^2}{\rho} = \frac{(\rho\Psi)^2}{\rho} = \rho\dot{\Psi}^2$ (8)

The above equation indicates that acceleration is function of time to get the acceleration as function of S by the following:

Since $s = 4a\sin\Psi$ (9)

By differentiating with respect to Ψ, we can get

$\frac{ds}{d\Psi} = 4a\cos\Psi$ (10)

From equations (4), (6), (9)

$a = \frac{V^2}{\rho} = \frac{V^2}{4a\cos\Psi} = \frac{V^2}{\left[4a\sqrt{1-\sin\Psi}\right]} = \frac{V^2}{\left[4a\sqrt{1-\left(\frac{s}{4a}\right)^2}\right]}$ (11)

3) A particle moves in a curve (S = C tan ψ), the direction of its acceleration at any point makes equal angles with the tangent and the normal to the path if the speed at ψ=0 be v= u show that the velocity and the acceleration at any other point are given by

$$\mathbf{v} = ue^{\Psi} \qquad\qquad a = \frac{u^2}{\sqrt{2}c}e^{2\Psi}\cos^2\Psi$$

Solution

Since acceleration at any point makes equal angles with the tangent and the normal to the path so

$a_r = a_n$ (1)

So $\ddot{s} = \frac{v^2}{\rho}$ (2)

And $\ddot{s} = V\frac{dV}{ds}$, $\rho = \frac{ds}{d\Psi}$

So $V\frac{dV}{ds} = d\Psi\frac{V^2}{ds}$ (3)

So

$\int\frac{dV}{\text{v}} = \int d\Psi$ (4)

By integrating both side the following form can get

$\ln \text{v} = \Psi + \text{c}$ (5)

Since ψ=0 be v= u so c=$\ln \text{u}$

$\ln \text{v} = \Psi + \ln \text{u}$ (6)

So $\mathbf{v} = ue^{\Psi}$ (7)

Since $a = \sqrt{(a_r)^2 + (a_n)^2}$ (8)

From equations (1) and (8) get:

Since $a = \sqrt{2}(a_n)$ (9)

Where $a_n = \frac{v^2}{\rho}$ (10)

Since $s = c\tan\Psi$ (11)

By differentiating with respect to Ψ, we can get

$\frac{ds}{d\Psi} = c\sec^2\Psi$ (12)

Since $\rho = \frac{ds}{d\Psi}$ (13)

So $\rho = c\sec^2\Psi$ (14)

From equations (7) and (14) get

Where $a_n = \frac{u^2 e^{2\Psi}}{c \sec^2 \Psi} = \frac{u^2}{c} e^{2\Psi} \cos^2 \Psi$ (15)

From equations (9) and (15) get

$$a = \frac{u^2}{\sqrt{2}c} e^{2\Psi} \cos^2 \Psi$$

4) Velocity of a moving point parallel to the axes of x and y are $\mathrm{u} + e^{\mathrm{y}}$ and $\mathrm{v} + e^{\mathrm{x}}$ respectively; show that the path is a conic section

Solution

Since

$$\frac{dx}{dt} = \mathrm{u} + e^{\mathrm{y}} \qquad (1)$$

And $$\frac{dy}{dt} = \mathrm{v} + e^{\mathrm{x}} \qquad (2)$$

Dividing equation (2) by equation (1) get:

$$\frac{dy}{dx} = \frac{\mathrm{v}+e^{\mathrm{x}}}{\mathrm{u}+e^{\mathrm{y}}} \qquad (3)$$

Separating variables can get

$$(\mathrm{u} + e^{\mathrm{y}})dy = (\mathrm{v} + e^{\mathrm{x}})dx \qquad (7)$$

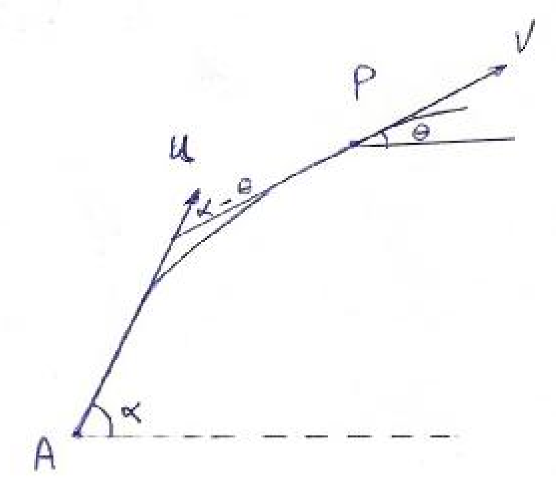
By integrating both side the following form can get

$(\mathrm{u} + e^{\mathrm{y}})^2 = (\mathrm{v} + e^{\mathrm{x}})^2 + c$ The path is a **conic section**

Chapter 2 - Projectieles

(**1**) If at any instant the velocity of the projectile be u and its direction of motion to the horizon is α Then it will be moving at right angles to this direction after time u/g*cosec α

Solution



Let the velocity at any point be v at an angle ϴ to the horizon then

V cos ϴ = u cos α

sin ϴ = u sin α

an ϴ =(u sin α - g* t)/(u cos α)

The angle between the two direction is (α-ϴ)

The condition of moving at right angles is

α –ϴ =π/2 or ϴ=α-π/2

hence

(- cot α) = (u sin α –g*t)/ u cosα

=tan α-(g*t)/u cosα

So g*t/(cos α) = tan α+ cot α

= (1+(tan α)^2)/ tan α

= (sec α)^2 /tan α

u/g * sec α/tan α = t

u/g * sin α

2- Three particles are simultaneously projected in the same vertical planefrom the same point with velocities $\boldsymbol{u}$ **,** $\boldsymbol{v}$**,** $\boldsymbol{w}$ at angles $\boldsymbol{\alpha}$**,**$\boldsymbol{\beta}$**,**$\boldsymbol{\gamma}$ with the horizontal .

Show that the area of the triangle formed by the particles at any time t is proportional to the square of the time elapsed from the instant of projection and that the three particles will always lie in the same straight line if

$$Sin(\beta-\gamma)/ u + \sin(\gamma-\alpha)/ v + \sin(\alpha -\beta) / w = 0$$

<u>Solution :</u>

If (x1,y1) ,(x2,y2) ,(x3,y3) be the positions of the particles at time t then

X1= (u cos α)* t ,y1 =(u sin α) *t -0.5 * g*t^2

X2= (u cos β)* t ,y2 =(u sin β) *t -0.5 * g*t^2

X3 = (u cos γ)* t y3 =(u sin γ) *t -0.5 * g*t^2

Area of the triangle

=0.5 *(x1(y2-y3) +x2(y3-y1) +x3(y1-y2))

=0.5 *(u cos α* t(v sin β – w sin γ) *t +(v cos β * t(w sin γ –u sin α)*t +(w cos γ*t)(u sin α-v sin β)*t)

0.5 t^2(u v (cos α sin β –sin α cos β) + v w (sin γ)+ u w (sin α cos γ –cos α sin γ))

= cos β – sin β cos γ

=0.5 t^2 (u v sin (β –α) +v w sin (γ – β) +u w sin (α –γ))

So the area of the triangle is proportional to the square of the time elapsed from the instant of projection

If the particles are in straight line the area vanishes hence it is the result required

3- A stone is projected with velocity u from a height h to hit a point in the level at the horizontal distance R from the point of the projection is given by

$$R^2 (\tan\alpha)^2 - 2 u^2/g * R \tan\alpha + R^2 - 2 h u^2/g = 0$$

Hence deduce that the maximum range on the level for this velocity is

$$((u^4/g^2) + (2 h u^2/g))^{0.5}$$

Solution

The point (R, -h) is on

$y = x \tan\alpha - 0.5 (g x^2)/(u^2 * (\cos\alpha)^2)$

$= -h = R \tan\alpha - 0.5 (g R^2)/(u^2 * (\cos\alpha)^2)$

$= R \tan\alpha - 0.5 (g R^2) * (1+(\tan\alpha)^2) / (u^2$

Or

$R^2 (\tan\alpha)^2 - 2 u^2/g * R \tan\alpha + R^2 - 2 h u^2/g = 0$ (1)

R is maximum when $(dR/d\alpha) = 0$

$2 R^2 \tan\alpha * (\sec\alpha)^2 - 2 u^2/g * R(\sec\alpha)^2 = 0$

Giving

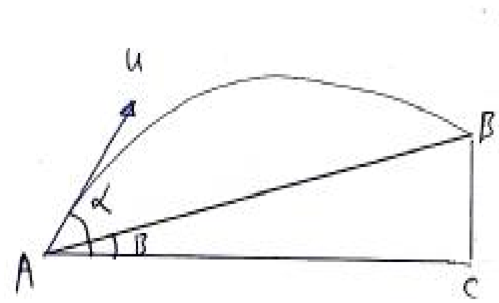
$R = u^2/g \cot\alpha$ or $\tan\alpha = u^2/gR$

SUBSTITUTING THIS VALUE OF R IN (1) WE GET

$R = ((u^4/g^2) + (2 h u^2/g))^{0.5}$

4- an enemys position on the hill h meters high is ar an angle of elevation of β . show that the least velocity of projection to shell the enemys position is(g h(1 +cosec β))^0.5

Solution



Let u be the least velocity

Range = h cosec β

H cosec β should be the maximum range

h cosec β =u^2/g * (1+ sin β)

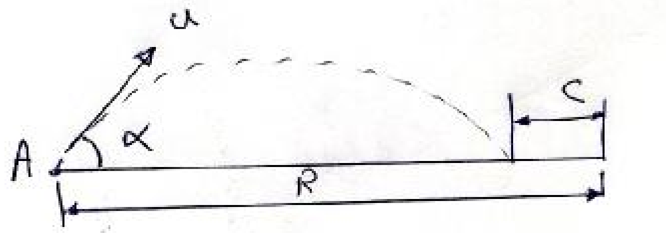
u^2 = h cosec β g (1+sin β)

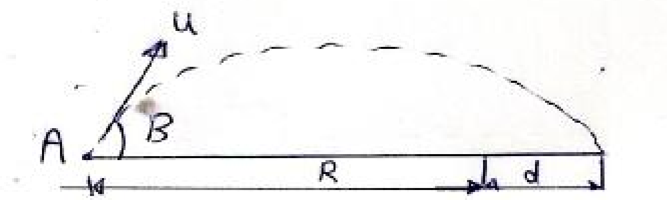
u = (g h(1 +cosec β))^0.5

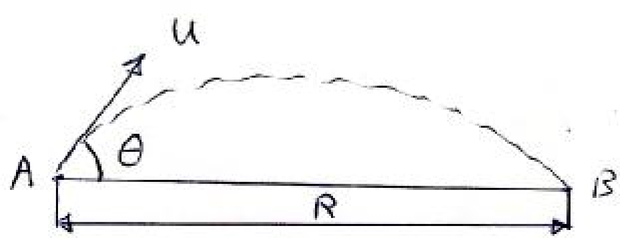
which is wanted

5- if the particle is projected at angle α to the horizon from the foot of an inclind plane whose inclination to the horizon is β strikes the plane at right angles show that
Cot β = 2 $\tan(\alpha-\beta)$

<u>Solution</u>







Let u be the velocity of projection

Initial velocity along the plane

$=u \cos (\alpha - \beta)$

Initial velocity normal to the plane

$= u \sin (\alpha - \beta)$

Acceleration due to gravity along the plane

$= g \sin \beta$

Acceleration due to gravity normal the plane

$=g \cos \beta$

Let T be the time of flight to strike the plane

For the motion normal to the plane , since the net distance moved is zero we have

$0 = u \sin (\alpha - \beta) T - 0.5 g \cos \beta T^2$

$T = 2 u (\sin (\alpha - \beta))/ (g \cos \beta)$

Now consider the motion along the plane since the particle strikes the plane at B at right angles to it the component of velocity along the plane is at B is zero

$0= u \cos (\alpha - \beta) - g \sin \beta T$

$0= u \cos (\alpha - \beta) - g \sin \beta * (2 u \sin (\alpha - \beta)/ g \cos \beta)$

$2 \sin (\alpha - \beta) \tan \beta = \cos(\alpha - \beta)$

$\text{Cot } B = 2 \tan (\alpha - \beta)$

6- Two particles are simultaneously projected in the same vertical plane from the same point with velocities u and v at angles α and β with the horizontal show that

a- the line joining them moves parallel to it self .

b- the time that elapses when their velocities are parallel is

$$\frac{u\,v\sin(\alpha-\beta)}{g(v\cos\beta-u\;\cos\alpha\,)}$$

c- the time that elapses between their transists through the other common point is

$$\frac{2\,u\,v\sin(\alpha-\beta)}{g(u\cos\alpha+v\cos\beta\,)}$$

<u>Solution</u>

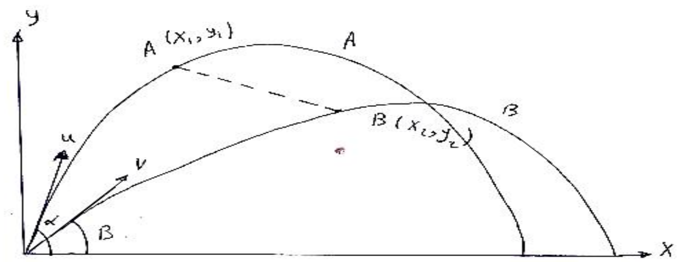
let after time t the two particles reach A and B having coordinate (x1,y1) , (x2,y2)

for the particle A

y1 = u sinα t – 0.5 g * t^2

and

x1 = u cos α t



for the particle B

y2 = V sinβ t – 0.5 g * t^2

and

x2 = V cos β t

AND THE SLOPE OF THE LINE

= (y2-y1)/(x1-x2)

$$=\frac{-(\mathrm{u\ sin\alpha\,t-\ 0.5\,g\ *\ t^2})+(\mathrm{V\ sin\beta\,t-\ 0.5\,g\ *\ t\hat{}2})}{\mathrm{u\ cos\,\alpha\,t-\ V\cos\beta\,t}}$$

= constant$\frac{u\ \sin\alpha-v\sin\ \beta}{v\cos\ \beta-u\cos\ \alpha}$=

So the slope is constant so the line joining them moves parallel to it self

b- Now consider the first particle . suppose its coordinates at any time t be (x,y) , we have

y = u sinα t – 0.5 g * t^2
and

$x = u \cos\alpha\, t$

$dy/dt = u \sin\alpha - g\,t$ and $dx/dt = u\cos\alpha$
so $dy / dx = (dy/dt)/(dx/dt) = (u \sin\alpha - g t)/(u\cos\alpha)$

but dy /dx represent the slope of direction at any time t

similarly the slope of the second particle should be the same we have

$=(u \sin\alpha - g t)/(u \cos\alpha) = (v \sin\beta - g t)/(v \cos\beta)$

$=(u \sin\alpha - g t(*(v \cos\beta) = (v \sin\beta - g t) *(u \cos\alpha)$

So $t\,g(v \cos\beta - u \cos\alpha) = u\,v\,(\sin\alpha \cos\beta - \sin\beta \cos\alpha) = u\,v \sin(\alpha - \beta)$

$$t = \frac{u\ v \sin(\alpha-\beta)}{g(\ v\cos\beta - u\cos\alpha}$$

c- The equation of path of the first particle is

$y = x \tan\alpha - (g x^2) /(2 u^2(\cos\alpha)^2)$

the equation of path of the second particle is

$y = x \tan\beta - (g x^2) /(2 v^2(\cos\beta)^2)$

Corresponding to the point of intersection of the two paths we have

$x \tan\alpha - (g x^2) /(2 u^2(\cos\alpha)^2) = x \tan\beta - (g x^2) /(2 v^2(\cos\beta)^2)$

$x(\tan\alpha - \tan\beta) = g x^2 /2 *\{1/ u^2(\cos\alpha)^2 \ - 1/ v^2(\cos\beta)^2\}$

$= g x /2 *\{(v^2(\cos\beta)^2 - u^2(\cos\alpha)^2)/ (u^2(\cos\alpha)^2 * v^2(\cos\beta)^2\}$

The x-coordinates of the point of intersection C

$$X = \frac{2(\tan\alpha - \tan\beta)\ v^2\ \cos\beta^2\ \cos\alpha^2}{g(\ v^2\cos\beta^2 - u^2\cos\alpha^2)}$$

Time taken by the first particle to reach c

$T1 = x/ u\cos\alpha$

$$= \frac{2(tan\alpha - tan\beta)v^2cos\beta^2\ u\ cos\alpha}{g(v^2cos\beta^2 - u^2cos\alpha^2)}$$

Similarly time taken by the second particle to reach c

$T2 = x / v\cos\beta$

$$= \frac{2(tan\alpha - tan\beta)u^2cos\alpha^2\ v\ cos\beta}{g(v^2cos\beta^2 - u^2cos\alpha^2)}$$

The time that elapses between their transits through the common c

$$\frac{2\ u\ v\ (\tan\alpha - \tan\beta)\cos\alpha\cos\beta}{g(v\cos\beta + u\cos\alpha)}$$

Chapter 3 - Simple harmonic motion

(1) A point moving with S .H.M , has a velocity of 4 feet/sec when passing through the center of its path , and its period is Л seconds ; what is its velocity when it has described one foot from the position in which its velocity is zero?

Solution.

The velocity at the center: wa

The periodic time = 2π /w

Substitute 2 π /w = π and wa = 4

Then, w = 2 and a = 2 feet

The velocity at distance x from the centre is

$$V = w\sqrt{a^2 - x^2} = 2\sqrt{4 - x^2}$$

When

$$X=1, \quad V = 2\sqrt{3} \quad \text{ft/sec}$$

2) A horizontal shelf moves vertically with S .H.M, whose complete period is one second. Find the greatest amplitude in centimeters that it can have, so that object resting on the shelf may always remain in contact with it.

Solution

The periodic time $2\pi / w = 1$ $w = 2\pi$

When the amplitude is a , then the maximum acceleration

$$W^2 a = 4\pi^2 a$$

Then the least reaction $R = m(g\ w^2 a)$

Where g is the acceleration of gravity.

The objects on the shelf may always remain in contact

When $R = 0$

I.e .

$$A - w^2 a > 0$$

Or

$$a < \alpha / w^2$$

i.e .

$$\alpha < 8 / \pi^2$$

And the periodic time

$$\tau = \pi \sqrt{ml/\lambda}$$

3) An elastic string, to the middle point of which a particle is attached, is stretched to twice its natural length and placed on a smooth horizontal table, and its ends are then displaced in the direction of the string find the period of oscillation

Solution.

If it is the natural length, λ is the modulus of elasticity.

If the particle is displaced a distance x then the equation of motion is

$$m\ddot{x} = -\frac{\lambda}{\ell/2}(x + l - l/2) + \frac{\lambda}{\ell/2}(l - x - l/2)$$

I,e $$m\ddot{x} = \frac{4\lambda}{m\ell} x$$

Or $$\ddot{x} = -w x$$

$$2\pi^2/g(47.6875) = (11 * 77)/(21 * 21)$$

And then $g = 981$ cm/sec^2

4) A body is attached to one end of an inextensible string and the other end moves in a vertical straight line with n complete oscillations per second . Show that the string will not remain tight during the motion unless.

$$n^2 < g/(4 \pi^2 a)$$

Where a is the amplitude of the motion.

Solution

The maximum acceleration of the upper and executing S.H.M . is ua and its period is

$$T = \pi / \sqrt{\mu}$$

Also

$$T = 1/n$$

Hence

$$U = 4 \pi^2/T^2 = 4 \pi n^2$$

So that the maximum acc. $= 4 \pi^2 n^2 a$

The maximum acc . of the particle is g which is possible when the string is not tight hence the string will not remain tight if the acc. Of the upper end is greater than g

i.e , the string will not remain tight unless

$$n^2 < g/4 \pi^2 a$$

5) A particle m is attached to a light wife which is stretched tightly between tow fixed points with a tension T. if a , b are distances of the particle from the two ends , prove that the period of a small transverse oscillation of m is

$$2\,\pi\sqrt{\frac{mab}{T\,(a+b)}}$$

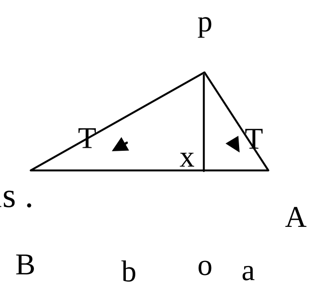
<u>Solution</u>

When the particle is slightly displaced at right angles to AB and it is a position P at time t where OP = x, the tension T remains practically the same.

Sum of components of

Two t,s along po

$$= T\sqrt{\frac{x}{a^2+x^2}} \quad + T.\sqrt{\frac{x}{b^2+x^2}}$$



= T(x/a + x/b) neglecting other terms .

Hence the equation of motion is

$$\ddot{M}x = Tx(1/a + l/b) = -T\ a + b/\ ab\ x$$

i.e , $$\ddot{x} = \frac{T(a+b)}{mab} \quad x$$

this is a S.H.M of periodic time $=\sqrt{2\,\pi}$ mab/t(a+b) if the length of the wire be given , i.e. , a + b is given , then time period is maximum when ab is maximum , i.e , when a = b .

Hence for a wire of given length the period is longest when the particle is attached to the middle points.

6) A heavy particle is attached at one points of a uniform light elastic , the ends of the string are attached to two points in a vertical line .show that the period of vertical oscillation in which the string remains taut is $2\pi\sqrt{(mh/x\pi)}$, where λ is the coefficient of elasticity of the string and h the harmonic mean of the unscratched length or the two parts of the string .

<u>Solution</u>

O, B are the points where the ends are attached and A the points where the particle is attached . let OA = a, AB = b and let / i1, i2 be natural , lengths of these two portions . Therefore in equilibrium

$$\lambda\,(a- l_1)/ l_1 = mg + \lambda(b- l_2)/ l_2 \qquad \ldots\ldots..(1)$$

Let the particle be slightly displaced when it is at P where AP = x , the lengths of the two portions are a + x , b –x and T1 and T2 be their tensions

$$T_1 = \lambda \frac{a + x - \ell_1}{\ell_1} \qquad T2 = \lambda \frac{B - x- \ell_2}{\ell_2}$$

Therefore the equation of motion is

$$mx = mg + T_2 + T_1 = mg + \lambda \frac{b - x- \ell_2}{\ell_2} - \lambda \frac{\lambda\, a + x - \ell_1}{\ell_1}$$

$$= \lambda x(l/l_1 + l/l_2) = - \lambda x\,[(l_1 + l_2) / (l_2\, l_1)\,]$$

$$= - (2\,\lambda/h)\,x$$

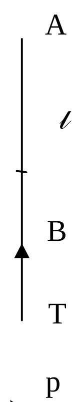
$$\therefore\ \text{Period} = 2\pi\sqrt{mh/ 2\,\lambda}$$

7) A tight elastic string of natural length l has one extremity fixed at a point A and the other attached to a stone the weight of which in equilibrium , would extend the string to a length l_1; show that if the stone be dropped from rest at A it will come to instantaneous rest at a depth $\sqrt{l_1^2 - l^2}$ below the equilibrium position and this depth attained in time $\sqrt{2\,l/g} + \sqrt{(l_1 - l)/g}$ $[\pi - \cos^{-1}\sqrt{(l_1 - l)/(l_1 + l)}\,]$

Solution

Let P be the position of the particle at time t , where AP = x .

The equation of motion is



$$m\ddot{x} = mg - T$$

$$= mg - \lambda\,(x - l)/l$$

In equilibrium position

$$\ddot{x} = 0,\ x = l_1$$

$$\therefore\ 0 = mg - \lambda\,(l_1 - l)/l$$

Hence $\quad m\ddot{x} = mg - (mg/(l_1 - l)).(x - l) = -(mg/(l_1 - l))\,.(x - l_1)$

i.e.,

$$\ddot{x} = (g/(l_1 - l))\,(x - l_1)$$

Integrating this equation we get

$$v^2 = (g/(l_1 - l))\,(x - l_1)^2 + c$$

Where c is constant

Now the simple harmonic motion begins when the particle has fallen freely through a distance $l_1$so that when $x = l$, $v^2 = 2g\,l$

$$2\,g\,l_1 = (g/(l_1 - l))\,(l - l_1) + c$$

$$c = 2g\,l_1 - g(l - l_1) = g\,(l + l_1)$$

Hence

$$v^2 = g(l-l_1)-(g/(l-l_1))\,(x-l)^2 \quad \ldots(1)$$

The particle will come to rest when x = o

$$(x-l_1)^2 = (l_1^2-l^2)$$

I.e.

$$x - l_1 = \sqrt{(l_1^2-l^2)}$$, which is the answer .

Time for free fall is given by $l = 1/2\, g\, t^2$

i.e. ,

$$t = \sqrt{2l/g}$$

from (1) $v^2 = g/(l-l_1)\,[(l_1^2-l^2)-(x-l_1)^2]$

time from B to the lowest point is given by

$$\sqrt{(g/(l-l_1))}\; t = \int_{\ell}^{\sqrt{\ell+\ (\ell_1^2-\ell^2)}} \frac{dx}{[(\ell_1^2-\ell^2)-(x-\ell_1)^2]} = \left[\sin^{-1}\frac{x-\ell_1}{(\ell_1^2-\ell^2)}\right]_{\ell}^{\ell+\sqrt{(\ell_1^2-\ell^2)}}$$

$$= \sin^{-1}1 + \sin^{-1}[(l_1-l)/(l_1+l)]$$

$$= \pi/2 + \pi/2 - \cos^{-1}[(l_1-l)/(l_1+l)] = \pi - \cos^{-1}[(l_1-l)/(l_1+l)]$$

Total time of fall

$$\sqrt{2l/g} + \sqrt{(l_1-l)}\,/g\ \ [\pi - \cos^{-1}[(l_1-l)/(l_1+l)]]$$

Further if the greatest depth below the centre be $l \cot^2 \theta/2$

We have

$$l \cot^2 \theta/2 = l_1 + \sqrt{l_1^2 - l^2}$$

Or

$$l_1^2 - l^2 = l^2 \cot^4 \theta/2 + l_1^2 - 2\,l\,l_1 \cot^2 \theta/2$$

$$l_1 = l/2\ \ \frac{L + \cot^4 \theta/2}{\mathrm{Cot}^2\,\theta/2}$$

Hence

$$\lambda = \frac{mg\, l_1^2}{l_1 - l} = \frac{2mg \cot^2 \theta/2}{(l - \cot^2 \theta/2)^2} = \quad 1/2\ mg \tan^2 \theta$$

also time of fall =

$$\sqrt{\frac{2\,l}{g}}\ [\,1 + \frac{l_1 - l}{2\,l}\ \left(\pi - \cos^{-1}\ \sqrt{\frac{l_1 - l}{l_1 + l}}\right)]$$

Now

$$\sqrt{\frac{l_1 - l}{2\,l}} = \frac{\text{Cot}^2\ \theta/2 - 1}{2 \cot \theta/2} \quad = \cot \theta$$

And

$$\sqrt{\frac{l_1 - l}{l_1 + l}} = \frac{(\text{Cot}^2\ \theta/2 - 1)^2}{2 \cot \theta/2 + 1} = \cos \theta$$

Hence time = 2 l /g   [l_1 + (π- θ) cot θ]

Chapter 4- Work & Energy

1) Find the work done by the force $\boldsymbol{F=[(6x^2+2xy^2)i+(2x^2y+5)j]}$ N. along

a) the line segment (0,0) to (1,3)

b) the part of the parabola $\boldsymbol{y=6x^2}$ from (0,0) to (1,3)

Solution

a- the work done along the line segment (0,0) to (1,3)

$$W=\int_{0,0}^{1,3}F.dr=\int_{0,0}^{1,3}(6x^2+2xy^2)dx+(2x^2y+5)dy=\int_0^1(6x^2+2xy^2)dx+\int_0^3(2x^2y+5)dy$$

$$W=\int_0^1(6x^2+2x(3x)^2)dx+\int_0^3\left(2\left(\frac{y}{3}\right)^2y+5\right)dy$$

$$W=\left|\left(\frac{6x^3}{3}+\frac{18x^4}{4}\right)\right|_0^1+\left|\left(\frac{y^4}{18}+5y\right)\right|_0^3=2+4.5+4.5+15=26\,joule$$

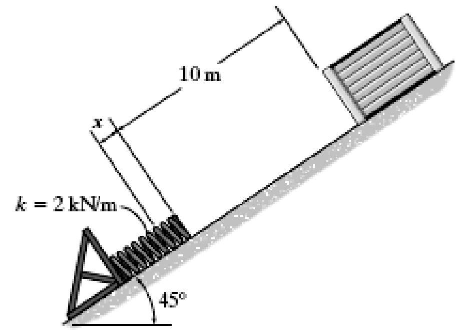
b- the work done along the parabola $\boldsymbol{y=6x^2}$ from (0,0) to (1,3)

$$W=\int_{0,0}^{1,3}F.dr=\int_{0,0}^{1,3}(6x^2+2xy^2)dx+(2x^2y+5)dy=\int_0^1(6x^2+2xy^2)dx+\int_0^3(2x^2y+5)dy$$

$$W=\int_0^1\left(6x^2+2x(3x^2)^2\right)dx+\int_0^3\left(2\left(\sqrt{\frac{y}{3}}\right)^2y+5\right)dy$$

$$W=\left|\left(\frac{6x^3}{3}+\frac{18x^6}{6}\right)\right|_0^1+\left|\left(\frac{3y^3}{9}+5y\right)\right|_0^3=2+3+6+15=26\,joule$$

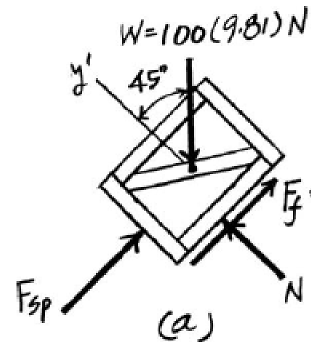
2) If the coefficient of kinetic friction between the **100**-kg crate and the plane is **μ=0.25**, determine the speed of the crate at the instant the compression of the spring is ***x = 1.5*** m. Initially the spring is unstretched and the crate is at **rest**.



Solution

$\Sigma F_y = 0$ N - 100(9.81)cos 45°=0

N = 693.67 *N*



Conservation of Energy:

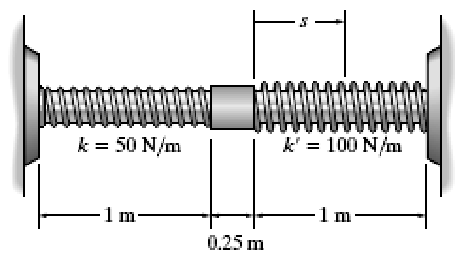
$(K_1 + U_1) = (K_2 + U_2)$

$\frac{1}{2}mv_1^2 + mgh = \frac{1}{2}mv_2^2 + \{mg(h) + \frac{1}{2}ks^2 + (mN)S\}$

0+100(9.8) (11.5 sin 45°) = 0.5(100) v^2 + 0 + 0.5(2000)$(1.5)^2$ + 0.25(693.67) 11.5

***v* =8.63 m/sec**

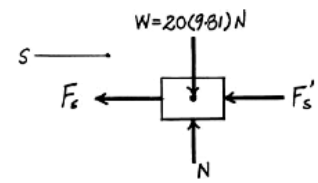
3) The collar has a mass of **20** kg and rests on the smooth rod. Two springs are attached to it and the ends of the rod as shown. Each spring has an uncompressed length of **1** m. If the collar is displaced **S=0.5** m and released from **rest**, determine its velocity at the instant it returns to the point **S = 0**.



Solution

Conservation of Energy:

$(K_1 + U_1) = (K_2 + U_2)$

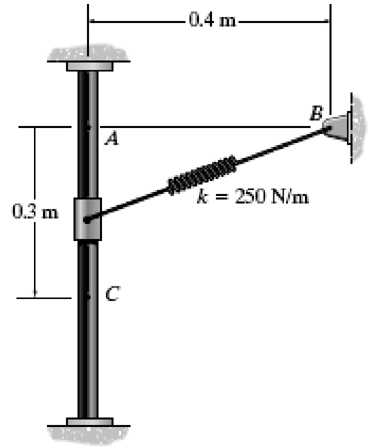


$\frac{1}{2}mv_1^2 + mgh + \frac{1}{2}k_1 s_1{}^2 + \frac{1}{2}k_2 s_2{}^2 = \frac{1}{2}mv_2^2 + mgh + \frac{1}{2}k_1 s_1{}^2 + \frac{1}{2}k_2 s_2{}^2$

0.5(20) v^2 + 0 + 0 +0 = 0 + 0 +0.5(50)(0.5)^2 +0.5(100)(0.5)^2

v =1.34 m/sec

4) The vertical guide is smooth and the **5-**kg collar is released from **rest** at **A**. Determine the speed of the collar when it is at position **C**. The spring has an unstretched length of **0.3**m.



Solution

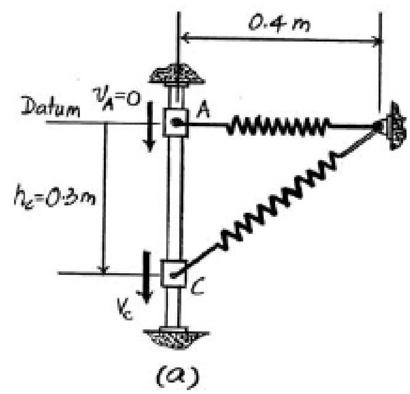
Conservation of Energy:

$$(K_A + U_A) = (K_C + U_C)$$

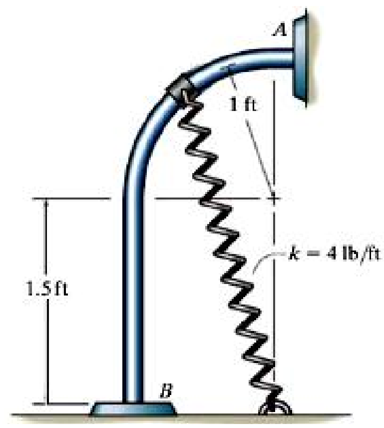
$$\tfrac{1}{2}mv_A^2 + mgh + \tfrac{1}{2}k_1 s_A{}^2 = \tfrac{1}{2}mv_c^2 + mgh + \tfrac{1}{2}k_1 s_c{}^2$$

$0 + 0 + 0.5(250)(0.1)^2 = 0.5(5)(vc)^2 - (5)9.8(0.3) + 0.5(250)(0.2)^2$

$\boldsymbol{v_c}$ **=2.1 m/sec**



5) The **5-**lb collar released from **rest** at **A** and travels along the frictionless guide. Determine the speed of the collar when it strikes the stop **B**. The spring has an unstretched length of **0.5** ft.



Solution

Conservation of Energy:

$$(K_A + U_A) = (K_C + U_C)$$

$$\tfrac{1}{2}mv_A^2 + mgh + \tfrac{1}{2}k_1 s_A{}^2 = \tfrac{1}{2}mv_B^2 + mgh + \tfrac{1}{2}k_1 s_B{}^2$$

$0 + 0 + 0.5(4)(2)^2 = 0.5(5/32.2)(v_B)^2 - (5)(2.5) + 0.5(4)(0.5)^2$

v_B =16 ft/sec

6) Determine whether or not the given force ***F= [(4xy) i + (2x²) j +4 k***] N is conservative, if so find its potential function, then calculate the work done by this force from point A= (2,4,-3)m to B=(1,-2,1) m.

Solution

Since $\frac{\partial Fx}{\partial y} = 4x$, $\frac{\partial Fy}{\partial x} = 4x$ and $\frac{\partial Fx}{\partial z} = 0$, $\frac{\partial Fz}{\partial x} = 0$ and $\frac{\partial Fy}{\partial z} = 0$, $\frac{\partial Fz}{\partial y} = 0$

Therefore this force is **conservative**.

To find the potential function $U(r)$, $U(r) = -\int_{\infty}^{r} \vec{F} . d\vec{r} = -\int_{\infty}^{r} \left[(4xy)dx + \left(2x^2\right)dy + (4)dz \right]$

$$U(r) = -\left[\left(2x^2 y\right) + \left(\cancel{2x^2 y}\right) + (4z) \right]$$

$$\therefore U(r) = -\left[\left(2x^2 y + 4z\right) + C \right]$$

work done from A to B = -[U_B-U_A]

$$\therefore U(A) = -\left[\left(2(2)^2(4) + 4(-3)\right) + C \right] = -[20 + C]$$

$$\therefore U(B) = -\left[\left(2(1)^2(-2) + 4(1)\right) + C \right] = -C$$

Therefore, the work done from A to B = -20 joule

7) Show that the force ***F* =[(2 y+2z²) *i* + (2x) *j* + (4xz) *k***] N is conservative, then find its potential function, and calculate the work done by this force along the line $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}=t,\ 0\le t\le 1$.

<u>Solution</u>

Since $\frac{\partial Fx}{\partial y}=2$, $\frac{\partial Fy}{\partial x}=2$ and $\frac{\partial Fx}{\partial z}=4z$, $\frac{\partial Fz}{\partial x}=4z$ and $\frac{\partial Fy}{\partial z}=0$, $\frac{\partial Fz}{\partial y}=0$

Therefore this force is **conservative**.

To find the potential function $U(r)$, $U(r)=-\int_{\infty}^{r}\vec{F}.\vec{dr}=-\int_{\infty}^{r}\left[\left(2y+2z^2\right)dx+(2x)dy+(4xz)dz\right]$

$$U=-\left[\left(2xy+2xz^2\right)+\cancel{(2xy)}+\cancel{\left(2xz^2\right)}\right]$$

$$\therefore U=-\left[\left(2xy+2xz^2\right)+C\right]$$

let point A lies on the curve at t=0 and point B at t=1

for point A (t=0) [x=0,y=0,z=0]

and for point B (t=1) [x=1,y=2,z=3]

$$\therefore U(A)=-\left[\left(2(0)(0)+2(0)(0)^2\right)+C\right]=-C$$

$$\therefore U(B)=-\left[\left(2(1)(2)+2(1)(3)^2\right)+C\right]=-[22+C]$$

Therefore, work done from A to B = -[U_B-U_A]= 22 joule

8) Find the conservative force which has the potential function ***u(x,y,z) = - (x³ y z²) + c*** ; then find the work done by this force from point A=(-1,-2,-1) to point B=(1,1,2).

Solution

Since $\vec{F} = -\left[\left(\frac{\partial u}{\partial x}\right)\vec{i} + \left(\frac{\partial u}{\partial y}\right)\vec{j} + \left(\frac{\partial u}{\partial z}\right)\vec{k}\right]$

$$\therefore \vec{F} = -\left[\left(\frac{\partial\left(x^3yz^2\right)}{\partial x}\right)\vec{i} + \left(\frac{\partial\left(x^3yz^2\right)}{\partial y}\right)\vec{j} + \left(\frac{\partial\left(x^3yz^2\right)}{\partial z}\right)\vec{k}\right]$$

$$\therefore \vec{F} = -\left[\left(3x^2yz^2\right)\vec{i} + \left(x^3z^2\right)\vec{j} + \left(2x^3yz\right)\vec{k}\right]$$

Work done from A to B = -[U_B-U_A]

$$\therefore U(A) = -\left[\left((-1)^3(-2)(-1)^2\right) + C\right] = -[2 + C]$$

$$\therefore U(B) = -\left[\left((1)^3(1)(2)^2\right) + C\right] = -[4 + C]$$

Therefore, the work done from A to B = 2 joule

9) Find the conservative force which has the potential function ***u(x,y,z) = - (xyz + 2y) + c***

Solution

Since $\vec{F} = -\left[\left(\frac{\partial u}{\partial x}\right)\vec{i} + \left(\frac{\partial u}{\partial y}\right)\vec{j} + \left(\frac{\partial u}{\partial z}\right)\vec{k}\right]$

$$\therefore \vec{F} = -\left[\left(\frac{\partial(xyz + 2y)}{\partial x}\right)\vec{i} + \left(\frac{\partial(xyz + 2y)}{\partial y}\right)\vec{j} + \left(\frac{\partial(xyz + 2y)}{\partial z}\right)\vec{k}\right]$$

$$\therefore \vec{F} = -\left[(yz)\,\vec{i} + (xz + 2)\,\vec{j} + (xy)\,\vec{k}\right]$$

Chapter 5- Motion in vertical circle

1) A body, of mass m, is attached to a fixed point by a string of length 3 feet ; it is held with the string horizontal and then let fall ; find its velocity when the string becomes vertical, and also the tension of the string then.

Solution:

The equation of energy is

$0.5\ mv2\text{-}mg\ \sin\alpha = 0$ i.e. $v2 = 2\ ga\ \sin\alpha$

Also, the equation of motion in the direction PO is

$V2/a = T/m - g\ \sin\alpha$ i.e. $T/m = 3\ g\ \sin\alpha$

Substitute for $a = 3$ ft , We find $\alpha = \pi/2$

$V = 8\sqrt{3}$ ft/sec. and $T/m = 3\ g$

2) Find the velocity with which a particle must be projected along the interior of a smooth vertical hoop of radius “r” from the lowest point in order that it may leave the hoop at an angular distance of 30° from the vertical. Show that it will strike the hoop again at an extremity of the horizontal diameter.

Show also that if velocity of projection be $\sqrt{\frac{7}{2}}$ gr, the particle will leave the hoop and return to the lowest point.

<u>Solution:</u>

The particle leaves the circle at an angle α from the upward drawn vertical where

Cos α = (1/3)(u2/gr – 2)

Hence

α = 300 $\sqrt{(\frac{3}{2})}$ = 1/3 (u2/gr – 2)

u2 = ½ gr (3 3+4)

since

BOQ = 300

BOD = 900

i.e. D is at the extremity of the horizontal diameter.

Where u2 = 7/2 gr

Cos α = 1/3 (7/2 - 2) =1/2

α = 600

BOQ = 600

Hence BOD = 1800

i.e. D coincides with A.

3) A particle is projected along the inside of a smooth vertical circle of radius r, from the lowest point. Show that the velocity of projection so that after leaving the circle the particle may pass through the center is $\sqrt{\frac{1}{2}gr\,(\sqrt{3}+1)}$

Solution:

The particle will leave the circle at some point Q at an angle α with the vertical with the velocity v.

Where

V2 = (u2 – 2gr)/3 = gr cos α

A cos α = 1/3 (u2/gr – 2)

With Q origin the center is the point (r sin α, -r cos α) and by the equation it lies on the parabola.

Y= x tan α -1/2 (gx2)/(v2 cos2α)

i.e. Y= x tan α -1/2 (gx2)/(gr cos3α)

Hence

-r cos α = r sin α tan α -1/2 (r2 sin2α)/(r cos3α)

i.e.

(sin2α / cos α) + cos α = ½ (sin2 α /cos3 α)

Or

Tan2α = 2

Sec2α = 3

Hence

Cos α =$\sqrt{(\frac{1}{3})}$

Therefore

$\sqrt{(\frac{1}{3})}$ = 1/3 (u2/gr – 2)

u2 = gr (2+3)

u = $\sqrt{(\frac{gr}{2})}(\sqrt{3}+1)$

4) A heavy particle hanging vertically from a fixed point by a light inextensible cord of length ℓ is struck by a horizontal blow which imparts to it a velocity 2gℓ; prove that the cord becomes slack when the particle has risen to a height 2/3 ℓ above the fixed point and find the highest point of the parabola subsequently described.

Solution:

The cord becomes slack at an angle α from the vertical where

$$\cos \propto = \frac{1}{3}\left(\frac{u^2}{gl} - 2\right) = \frac{1}{3}\left(\frac{4gl}{gl} - 2\right) = \frac{2}{3}$$

i.e. at a height above the fixed (center of the circle) = $l \cos \propto = \frac{2}{3} l$

the velocity at that point is v2 = $gl \cos \propto = \frac{2}{3} gl$

the greatest height above this point

$$= \frac{v^2 \sin^2 \propto}{2g} = \frac{\frac{2}{3}gl\left(1 - \frac{4}{g}\right)}{2g} = \frac{l}{3} \cdot \frac{5}{g} = \frac{5}{27} l$$

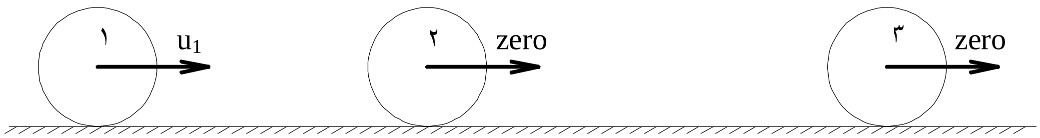
Chapter 6 - impact

1- *Three equal balls are on straight line on a table and one of them moves towards the other two which are at rest and not in contact, if e=1/2, find how many impacts will take place and show that the ultimate speeds of the balls are in the ratios 13:15:36.*

Solution:

Let, the three balls be 1, 2 and 3, the velocity of ball 1 is u_1.

Before any collision:



So, the first collision will be between ball 1 and 2.

Applying the law of conservation of momentum:

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

Therefore, $u_1 + 0 = v_1 + v_2$ (1)

Applying Newton experimental law:

$$\frac{v_1 - v_2}{u_1 - u_2} = -e$$

$$\frac{v_1 - v_2}{u_1 - 0} = -\frac{1}{2}$$

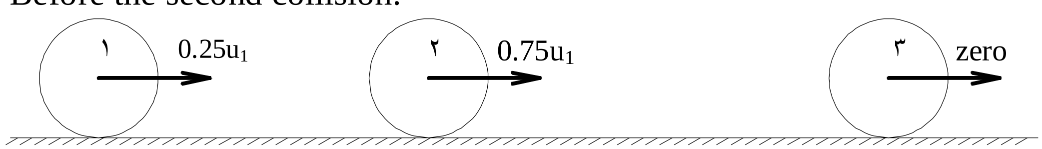
$(-0.5)u_1 = v_1 - v_2$ (2)

Adding equations (1) and (2)

$v_1 = 0.25u_1$

Substituting in equation (1) or (2) $v_2 = 0.75u_1$

Before the second collision:



The velocity of the second ball after collision is greater than that of the first ball, so there will be a second collision between ball 2 and 3.

Applying the law of conservation of momentum:

$0.75u_1 + 0 = v_3 + v_4$ (3)

Applying Newton experimental law:

$$\frac{v_3 - v_4}{0.75u_1 - 0} = -\frac{1}{2}$$

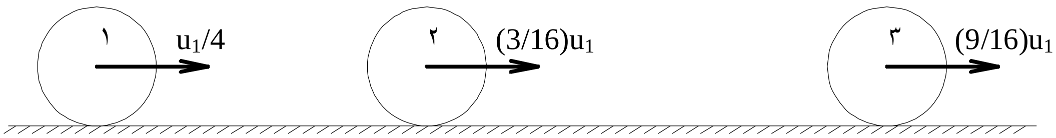
$$(-0.375)u_1 = v_3 - v_4 \qquad (4)$$

Adding equations (3) and (4)

v_3= (3/16) u_1 ((velocity of ball 2)

Substituting in equation (3) or (4) v_4= (9/16) u_1 (velocity of ball 3)

Before the third collision:



The velocity of the third ball after collision is greater than that of the second ball, and the velocity of the first ball is greater than the second one, so there will be another collision between ball 1 and 2.

Applying the law of conservation of momentum:

$$0.25u_1 + (\tfrac{3}{16})u_1 = v_5 + v_6 \qquad (5)$$

Applying Newton experimental law:

$$\frac{v_5 - v_6}{0.25u_1 - (\frac{3}{16})u_1} = -\frac{1}{2}$$

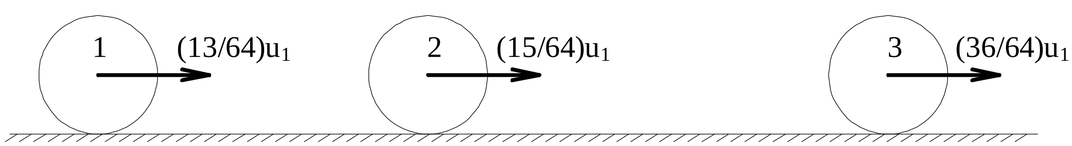
$$(-1/32)u_1 = v_5 - v_6 \qquad (6)$$

Adding equations (5) and (6)

v_5= (13/64) u_1 (velocity of ball 1)

Substituting in equation (5) or (6) v_6= (15/64) u_1 (velocity of ball 2)

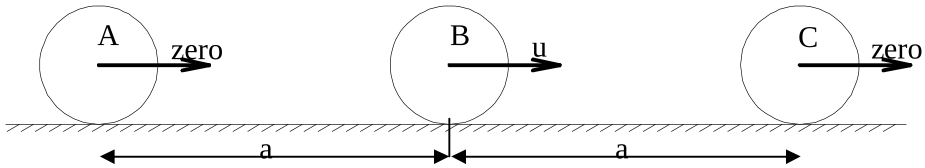
After the third collision:



The velocity of the second ball after collision is greater than that of the first ball, and the velocity of the third ball is greater than the second one, so there will be no more collisions, and it is clear from the results that the ratio between the velocities of the three balls are 13:15:36.

2- *Three small spheres A,B and C whose masses are 8m,m and 7m are at rest on a straight line where, AB=BC=a. The middle sphere B is projected toward C with velocity u. Assuming all spheres to be perfectly elastic, find the times and positions of impacts.*

Solution:



The first collision will be between B and C:

Applying the law of conservation of momentum:

$$mu + 0 = mv_B + 7mv_C$$

$$u = v_B + 7v_C \qquad (1)$$

Applying Newton`s experimental law:

$$\frac{v_B - v_C}{u - 0} = -e$$

Since, e=1

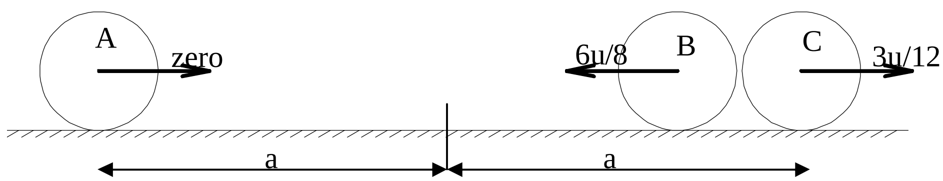
$$v_B - v_C = -u \qquad (2)$$

Subtract both equations (2) from (1)

$$v_C = \frac{3}{12}u$$

From equation (2)$v_B = \frac{-6}{8}u$, which means that the second collision will be between B and A.

The time at which the impact occur $t_1 = a/u$ (the velocity is uniform)



Applying the law of conservation of momentum:

$$m\frac{-6}{8}u + 0 = mv`_B + 8mv_A$$

$$\frac{-6}{8}u = v`_B + 8v_A \qquad (3)$$

Applying Newton`s experimental law:

$$\frac{v`_B - v_A}{\frac{-6}{8}u - 0} = -e$$

Since, e=1

$$v`_B - v_A = \frac{6}{8}u \qquad (4)$$

Subtract both equations (4) from (3)

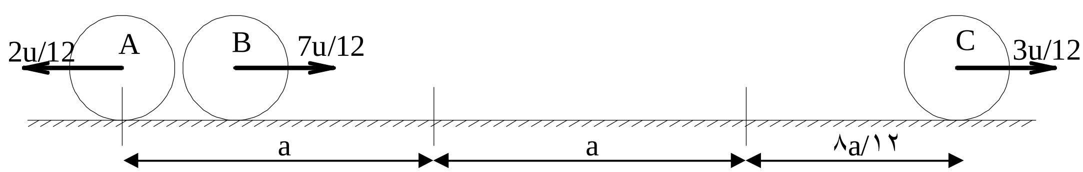
$$v_A = \frac{-12}{8*9}u = \frac{-2}{12}u$$

From equation (4) $v`_B = \frac{7}{12}u > v_c$ which means that the third collision will be between B and C.

The time ball B consumed to reach to ball A $t_2 = \frac{2a}{\frac{6u}{8}} = \frac{8a}{3u}$

The time measured from the initial position = $t_1 + t_2 = 11a/3u$

The distance covered by ball C=$\frac{3u}{12} * t_2 = \frac{3u}{12} * \frac{8a}{3u} = \frac{8a}{12}$



Applying the law of conservation of momentum:

$$m\frac{7}{12}u + 7m\frac{3}{12}u = mv``_B + 7mv``_C$$

$$\frac{28}{12}u = v``_B + 7v``_C \qquad (5)$$

Applying Newton`s experimental law:

$$\frac{v``_B - v``_C}{\frac{7}{12}u - \frac{3}{12}u} = -e$$

Since, e=1

$v``_B - v``_C = \frac{-4}{12}u$ (6)

Subtract both equations (6) from (5)

$v``_C = \frac{4}{12}u$

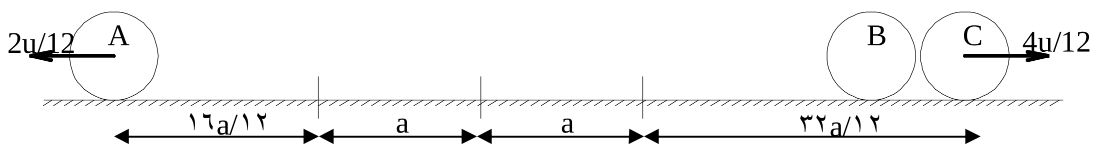
From equation (6)$v``_B = 0$ which means that there will be no more collisions.

The time covered by ball B to collide with C= t_3 = $\frac{2a+(\frac{8a}{12})}{(\frac{7u}{12})-(\frac{3u}{12})} = \frac{8a}{u}$

The time of the third impact measured from the initial position= $t_1+t_2+t_3$= $\frac{35a}{u}$

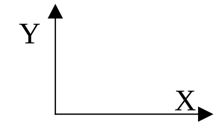
The distance covered by ball B to collide with C= $\frac{7u}{12} * t_3 = \frac{7u}{12} * \frac{8a}{u} = \frac{56a}{12}$

The distance covered by ball A at this time= $\frac{2u}{12} * t_3 = \frac{2u}{12} * \frac{8a}{u} = \frac{16a}{12}$



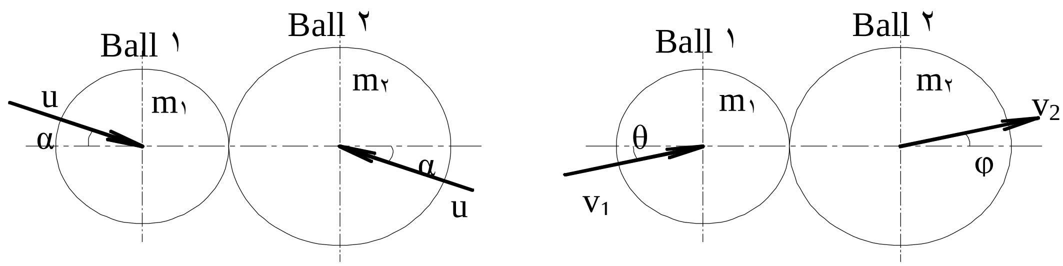
Note: In the last calculation of time (t_3) the relative velocity of B with respect to C is used, since the 2 balls is moving so to get the time at which B reaches C the relative velocity should be used (The relative velocity of B with respect to C is the velocity of B as if C is at rest).

3- *Two balls of elasticity e, moving in parallel directions with equal momentum impinge. Prove that if their directions of motion be opposite, they will move after impact in parallel directions with equal momentum.*



Solution:

Just before impact: Just after impact



Let, u_1, u_2 be the velocities of ball 1 and 2 respectively before impact, v_1, v_2 be their velocities after impact and m_1, m_2 be their masses.

Applying the law of conservation of momentum in the x direction:

$$m_1u_1cos\alpha + m_2(-u_2cos\alpha) = m_1v_1cos\theta + m_2v_2cos\varphi \qquad (1)$$

Since, $m_1u_1 = m_2v_2$ and u_1, u_2 are parallel

From (1) $0 = m_1v_1cos\theta + m_2v_2cos\varphi$

$$m_1v_1cos\theta = -m_2v_2cos\varphi \qquad (2)$$

Since, the two balls are smooth, therefore there will be no friction between them at the moment of impact, i.e. there will be no change in momentum in the y-direction

Therefore, $-m_1u_1sin\alpha = m_1v_1sin\theta \qquad (3)$

And $-m_2u_2sin\alpha = m_2v_2sin\varphi \qquad (4)$

Adding equations (3), (4) and using that $m_1u_1 = m_2v_2$

Therefore, $0 = m_1v_1sin\theta + m_2v_2sin\varphi$

$$m_1v_1sin\theta = -m_2v_2sin\varphi \qquad (5)$$

Dividing (2) and (5)

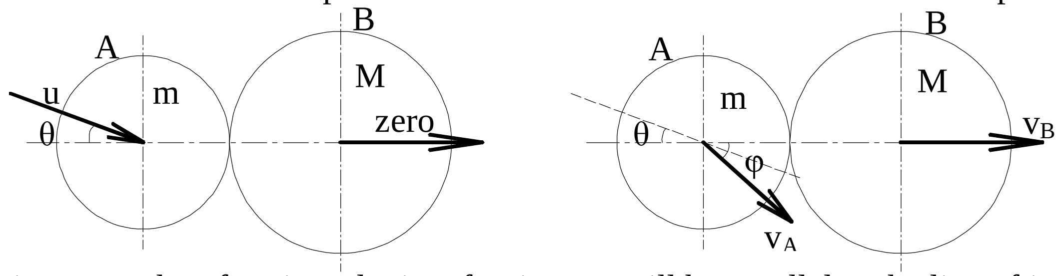
Therefore, $tan\theta = tan\varphi$ which implies that $\theta = \varphi$

From (5) or (2)

$m_1v_1 = m_2v_2$ i.e. v_1 and v_2 are parallel and in the same direction, which means that the two balls after impact move parallel to each other in the same direction.

4- *A smooth sphere of mass m travelling with velocity u impinges obliquely on a smooth sphere of mass M at rest; its original line of motion makes an angle θ with the line of centers at the moment of impact. Show that the sphere of mass m will be reflected at right angle if* $tan^2\theta = \frac{(eM-m)}{(M+m)}$.

Solution: Just before impact: Just after impact



Since ball B is at rest, therefore its velocity after impact will be parallel to the line of impact.

Applying the law of conservation of momentum in the x-direction:

$$mucos\theta + 0 = mv_A cos\varphi + Mv_B \qquad (1)$$

Applying Newton`s experimental law:

$$\frac{v_A cos\varphi - v_B}{ucos\theta} = -e$$

Therefore, $v_A cos\varphi - v_B = -eucos\theta$ (2)

Multiply equation (2) by M and add to (1), gives:

$$(m - eM)ucos\theta = (m + M)v_A cos\varphi \qquad (3)$$

Since, the two balls are smooth i.e. there is no friction, therefore there will be no impact perpendicular to the line of impact, so the components of the velocities perpendicular to the line of impact are conserved.

For ball A:

$$usin\theta = v_A sin\varphi \qquad (4)$$

If ball A is reflected through right angle i.e. $\varphi = \theta + 90$.

From equation (3)

$$(m - eM)ucos\theta = (m + M)v_A sin\theta \qquad (5)$$

From equation (4)

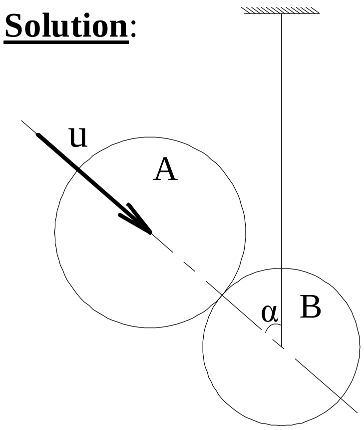
$$usin\theta = -v_A cos\theta \qquad (6)$$

Divide (5) and (6) gives:

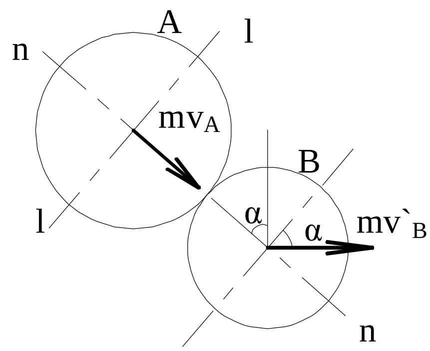
$(m - eM)cot\theta = -(m + M)tan\theta$ Therefore, $tan^2\theta = \frac{(eM-m)}{(m+M)}$

5- *A smooth spherical ball of mass m` is tied to a fixed point by a light inextensible string, and another spherical ball of mass m impinges directly on it with velocity u in a direction making an acute angle α with the string. Prove that the velocity with which the tied ball begins to move is* $\frac{m(1+e)u\sin\alpha}{m`+m\sin^2\alpha}$.

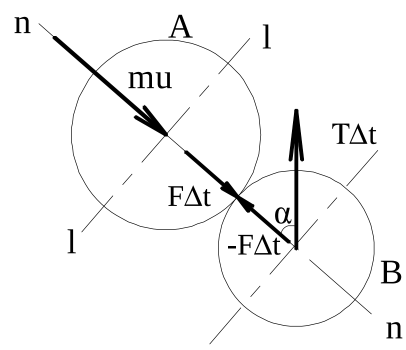
<u>Solution</u>:



Just after impact:



At the moment of impact:



It is clear that ball B can move only perpendicular to the string; therefore there will be no impact in the direction of the string.

Also, since there is no friction between the two balls (smooth balls) this means that there is no change in momentum perpendicular to n-n so the impinged ball will move on the same line after impact

Applying the law of conservation of momentum in the x-direction:

$$mu\sin\alpha = mv_A\sin\alpha + m`v_B \qquad (1)$$

Applying Newton`s experimental law between the two balls:

$$\frac{v_B\sin\alpha - v_A}{(0-u)} = -e$$

$$v_A = v_B\sin\alpha + eu \qquad (2)$$

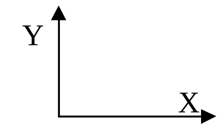
Substitute by (2) in (1)

$$mu\sin\alpha = mv_B\sin^2\alpha + meu\sin\alpha + m`v_B \qquad (3)$$

$$\therefore v_B = \frac{m(1+e)u\sin\alpha}{m`+m\sin^2\alpha}$$

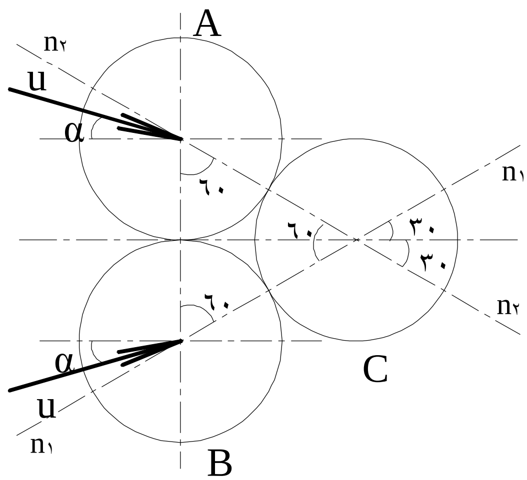
Hence the prove

6- *Two equal balls moving with equal velocities collide simultaneously with an equal ball at rest and with each other, their direction of motion being inclined at equal angle α to their common tangent at the moment of impact. Prove that after impact their direction is inclined at an angle* $2tan^{-1}\frac{3etan\alpha}{2-e}$ *to each other, and that the fraction* $\frac{(1-e^2)(1+2sin^2\alpha)}{3}$ *of the original kinetic energy disappears at after impact, where e is the coefficient of restitution between all the balls.*

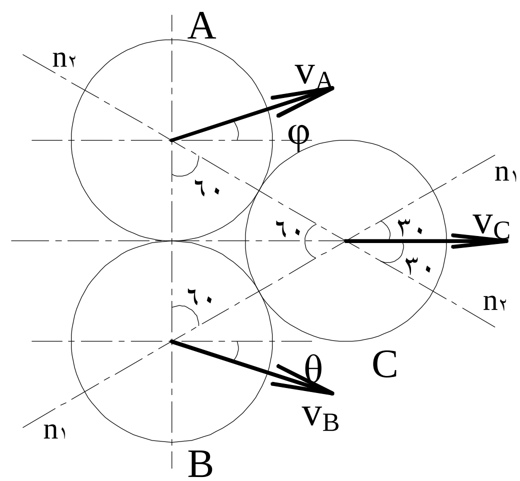


Solution:

Just before impact:



Just after impact:



Since the two balls A and B have the same mass and move with the same velocity and collides with ball Cat the same time with the same angle, therefore ball C will move after impact with a velocity parallel to the tangent of ball A and B i.e. the x-axis.

Also, because of the above the velocities of A and B after impact must be equal (v_A=v_B) and θ=φ).

To get the needed angle (2θ or 2φ)

Apply the law of conservation of momentum in the x-direction:

$$2mucos\alpha + 0 = mv_C + 2mv_A cos\varphi$$

$$v_C = 2ucos\alpha - 2v_A cos\varphi \qquad (1)$$

Applying Newton`s experimental law in direction n_1-n_1

$$\frac{v_A \cos(30+\varphi) - v_C cos30}{ucos(30-\alpha)} = -e$$

$$v_A \cos(30+\varphi) - v_C cos30 = -eucos(30-\alpha)$$

$$v_C = \frac{1}{cos30}(eucos(30-\alpha) + v_A \cos(30+\varphi))$$

$$v_C = \frac{1}{cos30}(eucos30cos\alpha + eusin30sin\alpha + v_A cos30cos\varphi - v_A sin30sin\varphi)$$

$$v_C = (eucos\alpha + \frac{eusin\alpha}{\sqrt{3}} + v_A cos\varphi - \frac{v_A sin\varphi}{\sqrt{3}}) \quad (2)$$

Subtract (2) from (1)

$$0 = (2u - eu)cos\alpha - 3v_A cos\varphi - \frac{eusin\alpha}{\sqrt{3}} + \frac{v_A sin\varphi}{\sqrt{3}} \quad (3)$$

Apply Newton`s Experimental law in direction y-direction

$$\frac{v_A sin\varphi - (-v_B sin\theta)}{-usin\alpha - usin\alpha} = -e$$

Since, $v_A = v_B$ and $\theta = \varphi$

Therefore, $2v_A sin\varphi = 2eusin\alpha$

$$v_A = \frac{eusin\alpha}{sin\varphi} \quad (4)$$

Substitute by (4) in (3)

$$0 = (2u - eu)cos\alpha - 3eusin\alpha cot\varphi - \frac{eusin\alpha}{\sqrt{3}} + \frac{eusin\alpha}{\sqrt{3}}$$

$$0 = (2u - eu)cos\alpha - 3eusin\alpha cot\varphi$$

$$cot\varphi = \frac{(2-e)}{3e} cot\alpha \quad (5)$$

$$tan\varphi = \frac{3e}{(2-e)} tan\alpha$$ Hence the prove

To get the fraction lost from the original kinetic energy after impact:

Get first v_c:

Substitute by (4) in (1)

$$v_C = 2ucos\alpha - 2eusin\alpha cot\varphi$$

$$v_C = 2ucos\alpha(1 - etan\alpha cot\varphi) \quad (6)$$

Substitute by (5) in (6)

$$v_C = 2ucos\alpha\left(1 - \frac{2-e}{3}\right) = \frac{2u(1+e)cos\alpha}{3} \quad (7)$$

Second kinetic energy before impact:

$$K.E._{before} = 2 * \frac{1}{2}mu^2 = mu^2 \quad (8)$$

Third kinetic energy after impact:

$$K.E._{after} = \frac{1}{2}mv_A^2 + \frac{1}{2}mv_B^2 + \frac{1}{2}mv_C^2$$

Since, $v_A = v_B$

Therefore, $K.E._{after} = mv_A^2 + \frac{1}{2}mv_C^2,$ (9)

Substitute by (4) and (7) in (9)

$$K.E._{after} = m(\left(\frac{eusin\alpha}{sin\varphi}\right)^2 + \frac{1}{2}\left(\frac{2u(1+e)cosa}{3}\right)^2)$$

$$K.E._{after} = mu^2(\frac{e^2(sin\alpha)^2}{(sin\varphi)^2} + \frac{2(1+e)^2(cosa)^2}{9})$$

$$K.E._{after} = mu^2(e^2(sin\alpha)^2(1+(cot\varphi)^2) + \frac{2(1+e)^2(1-(sin\alpha)^2)}{9})$$

$$K.E._{after} = mu^2(e^2(sin\alpha)^2(1+(\frac{(2-e)}{3e}cot\alpha)^2) + \frac{2(1+e)^2(1-(sin\alpha)^2)}{9})$$

$$K.E._{after} = mu^2(e^2(sin\alpha)^2 + \frac{(2-e)^2}{9}(cos\alpha)^2 + \frac{2(1+e)^2(1-(sin\alpha)^2)}{9})$$

$$K.E._{after} = mu^2(e^2(sin\alpha)^2 + \frac{(2-e)^2}{9}(1-(sin\alpha)^2) + \frac{2(1+e)^2(1-(sin\alpha)^2)}{9})$$

$$K.E._{after} = mu^2(e^2(sin\alpha)^2 + \frac{(2(1+e)^2+(2-e)^2)(1-(sin\alpha)^2)}{9})$$

$$K.E._{after} = mu^2(e^2(sin\alpha)^2 + \frac{3(2+e^2)(1-(sin\alpha)^2)}{9})$$

$$K.E._{after} = mu^2(\frac{9e^2(sin\alpha)^2 + 3(2+e^2) - 3(2+e^2)(sin\alpha)^2}{9})$$

$$K.E._{after} = mu^2(\frac{(6e^2-6)(sin\alpha)^2 + 3(2+e^2)}{9})$$

$$K.E._{after} = mu^2(\frac{(e^2-1)2(sin\alpha)^2 + (2+e^2)}{3})$$

$$K.E._{after} = mu^2(\frac{(2+e^2)-(1-e^2)2(sin\alpha)^2}{3}) \quad (10)$$

The fraction lost:

$$\frac{K.E._{loss}}{K.E._{before}} = \frac{K.E._{before} - K.E._{after}}{K.E._{before}} \quad (11)$$

Substitute by (8) and (10) in (11)

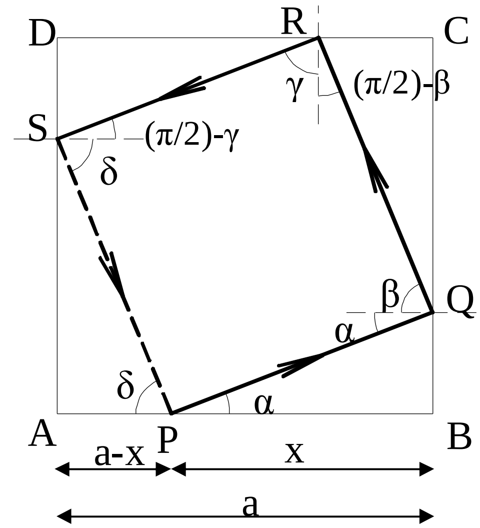
$$\frac{K.E._{loss}}{K.E._{before}} = 1 - \frac{(2+e^2)-(1-e^2)2(sin\alpha)^2}{3}$$

$$\frac{K.E._{loss}}{K.E._{before}} = \frac{(1-e^2)+(1-e^2)2(sin\alpha)^2}{3} = \frac{(1-e^2)(1+2(sin\alpha)^2)}{3}$$

Hence the prove

7- square table ABCD whose side is 'a' has raised edges, a particle of elasticity e is projected from a point P on AB and hits the sides BC, CD, and DA at Q, R, and S. Prove that PQ and RS are parallel. If α be the angle QPB and BP=x, prove that if the particle returns to P, $x(1-e)=a(1-e\cot\alpha)$.

Solution:



For PQ to be parallel to SR, angle α must equal angle (π/2)-γ, which can be easily proved using the relation between the angles of incident and reflection of the particle due to impact.

Studying the impact at Q:

$\tan\alpha = e\tan\beta$ (1)

Studying the impact at R:

$\tan\left(\frac{\pi}{2}-\beta\right) = e\tan\gamma$

$\cot\beta = e\tan\gamma$ (2)

From equation (1) and (2)

$\tan\alpha = \frac{e}{e\tan\gamma}$

Therefore, $\tan\alpha = \cot\gamma = \tan\left(\frac{\pi}{2}-\gamma\right)$ (3)

Therefore, α = (π/2)-γ Hence the prove

-Since, the particle returns again to point P so similarly as above QR must be parallel to SP, which implies that PQRS is a parallelogram.

Therefore, PQ=RS, and QR=SP

$\cot\alpha = \frac{x}{BQ}$ (4)

$\tan\delta = \frac{AS}{a-x}$ (5)

Since, PQ and RS are equal and parallel therefore, BQ=SD

From equation (5)

$$tan\delta = \frac{a-SD}{a-x} = \frac{a-BQ}{a-x} \tag{6}$$

Studying the impact at S

$$\tan\left(\frac{\pi}{2} - \gamma\right) = etan\delta$$

Therefore, $cot\gamma = etan\delta$ (7)

From (3) and (7)

$$tan\alpha = etan\delta \tag{8}$$

From (8), (6) and (4)

$$tan\alpha = e\frac{a - (\frac{x}{cot\alpha})}{a - x}$$

$$cot\alpha = \frac{a - x}{e(a - \left(\frac{x}{cot\alpha}\right))}$$

$$eacot\alpha - ex = a - x$$

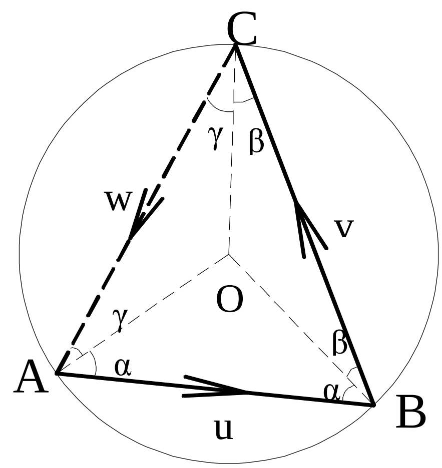
Therefore, $x(1 - e) = a(1 - ecot\alpha)$

Hence the prove

8- *A smooth circular table is surrounded by a smooth rim whose interior surface is vertical. Show that if a ball is projected along the table from a point on the rim in a direction making an angle α with the radius through the point it will return to the point of projection after:*

i) Two impacts if $\tan\alpha = \frac{e^{3/2}}{\sqrt{1+e+e^2}}$*, and the ratio between the velocity by which it returns and the velocity of projection is* $e^{3/2}$.

ii) Three impacts if $\tan\alpha = e^{3/2}$*, and that the angle between the direction of projection and the direction by which it returns is* 90^o.



Solution:

In case of two impacts as shown:

Assume the ball is projected fro point A with velocity u and hits the rim of the table at B at an angle of incidence α to the normal to the rim then it reflects with velocity v at an angle of reflection β to hit the rim at C at an angle of incidence β then reflects with velocity w and angle of reflection γ as shown.

It is clear that for the ball to return again to point A (point of projection) after two impacts the path of the ball must be a triangle.

For ABC to be a triangle the sum of its angle must be 180^o.

$$(\gamma + \alpha) + (\alpha + \beta) + (\beta + \gamma) = 180$$

$$\gamma + \alpha + \beta = 90$$

$$\tan\alpha = \tan(90 - (\gamma + \beta)) = \cot(\gamma + \beta)$$

$$\tan\alpha = \frac{1 - \tan\gamma\tan\beta}{\tan\gamma + \tan\beta} \qquad (1)$$

Study the impact at B and C to get the relation between the angles:

Study the impact at B:

$$\tan\alpha = e\tan\beta \qquad (2)$$

Study the impact at C:

$$tan\beta = etan\gamma \qquad (3)$$

From equation (2) and (3)

$$tan\alpha = e^2 tan\gamma \qquad (4)$$

Substitute by equation (2) and (3) in (1)

$$tan\alpha = \frac{1 - \frac{tan\alpha}{e^2}\frac{tan\alpha}{e}}{\frac{tan\alpha}{e^2} + \frac{tan\alpha}{e}} = \frac{e^3 - (tan\alpha)^2}{(e + e^2)tan\alpha}$$

$$(e + e^2)(tan\alpha)^2 = e^3 - (tan\alpha)^2$$

$$(tan\alpha)^2 = \frac{e^3}{(1 + e + e^2)}$$

Therefore, $tan\alpha = \frac{e^{3/2}}{\sqrt{1+e+e^2}}$ (5

Hence the prove

To get the ratio between the velocities:

Study the impact at B:

$$eucos\alpha = vcos\beta$$

$$u = \frac{vcos\beta}{ecos\alpha} \qquad (6)$$

Study the impact at B:

$$evcos\beta = wcos\gamma$$

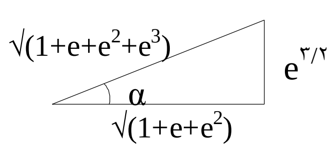
$$w = \frac{evcos\beta}{cos\gamma} \qquad (7)$$

From (6) and (7)

$$\frac{w}{u} = \frac{e^2 cos\alpha}{cos\gamma} \qquad (8)$$

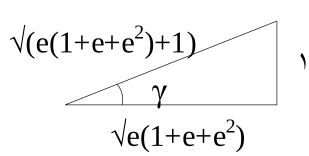
From (5)

$$cos\alpha = \sqrt{\frac{1+e+e^2}{1+e+e^2+e^3}} \qquad (9)$$



From (5) and (4)

$$tan\gamma = \frac{e^{\frac{3}{2}}/e^2}{\sqrt{1+e+e^2}} = \frac{1}{\sqrt{e(1+e+e^2)}}$$



Therefore, $cos\gamma = \sqrt{\frac{e(1+e+e^2)}{e(1+e+e^2)+1}}$ (10)

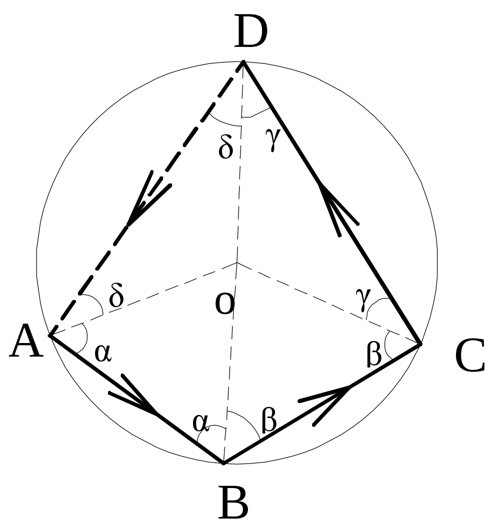
Substitute by (9) and (10) in (8)

$$\frac{w}{u} = e^2\sqrt{\frac{1+e+e^2}{1+e+e^2+e^3}}\sqrt{\frac{e(1+e+e^2)+1}{e(1+e+e^2)}}$$

$$\frac{w}{u} = e^2\sqrt{\frac{e+e^2+e^3+1}{e(1+e+e^2+e^3)}} = e^{3/2}$$

Hence the prove

In case of two impacts as shown:



Assume the ball is projected fro point A and hits the rim of the table at B, C and D to form the angles shown in figure.

It is clear that for the ball to return again to point A (point of projection) after three impacts the path of the ball must be a quadrilateral.

For ABCD to be a quadrilateral the sum of its angle must be 360°.

$$(\delta+\alpha)+(\alpha+\beta)+(\beta+\gamma)+(\gamma+\delta) = 360$$

$$\delta+\gamma+\alpha+\beta = 180$$

$$tan\alpha = \tan\big(180-(\delta+\gamma+\beta)\big) = -\tan(\delta+\gamma+\beta)$$

$$-tan\alpha = \frac{tan\delta+\tan(\gamma+\beta)}{1-tan\delta tan(\gamma+\beta)} \qquad (11)$$

$$tan(\gamma+\beta) = \frac{tan\gamma+tan\beta}{1-tan\gamma tan\beta} \qquad (12)$$

Study the impact at B, C and D to get the relation between the angles:

Study the impact at B:

$$tan\alpha = etan\beta \qquad (13)$$

Study the impact at C:

$$tan\beta = etan\gamma \qquad (14)$$

Study the impact at D:

$tan\gamma = etan\delta$ (15)

From equation (13), (14) and (15)

$tan\alpha = e^2 tan\gamma$ (16)

$tan\alpha = e^3 tan\delta$ (17)

Substitute by equation (13) and (16) in (12)

$$tan(\gamma+\beta) = \frac{\frac{tan\alpha}{e^2}+\frac{tan\alpha}{e}}{1-\frac{tan\alpha tan\alpha}{e^2\ \ e}} = \frac{(e+e^2)tan\alpha}{e^3-(tan\alpha)^2} \quad (18)$$

Substitute by equation (18) and (17) into (11)

First, $(tan\delta + \tan(\gamma+\beta)) = \frac{tan\alpha}{e^3} + \frac{(e+e^2)tan\alpha}{e^3-(tan\alpha)^2}$

$$(tan\delta + \tan(\gamma+\beta)) = \frac{e^3 tan\alpha - (tan\alpha)^3 + e^3(e+e^2)tan\alpha}{e^3(e^3-(tan\alpha)^2)}$$

$$(tan\delta + \tan(\gamma+\beta)) = \frac{e^3(1+e+e^2)tan\alpha - (tan\alpha)^3}{e^3(e^3-(tan\alpha)^2)} \quad (19)$$

Second, $1 - tan\delta tan(\gamma+\beta) = 1 - \frac{tan\alpha}{e^3} * \frac{(e+e^2)tan\alpha}{e^3-(tan\alpha)^2}$

$$1 - tan\delta tan(\gamma+\beta) = 1 - \frac{(e+e^2)(tan\alpha)^2}{e^3(e^3-(tan\alpha)^2)}$$

$$1 - tan\delta tan(\gamma+\beta) = \frac{e^3(e^3-(tan\alpha)^2) - (e+e^2)(tan\alpha)^2}{e^3(e^3-(tan\alpha)^2)}$$

$$1 - tan\delta tan(\gamma+\beta) = \frac{e^6 - (e^3+e+e^2)(tan\alpha)^2}{e^3(e^3-(tan\alpha)^2)} \quad (20)$$

Substitute by (19) and (20) in (11)

$$-tan\alpha = \frac{e^3(1+e+e^2)tan\alpha - (tan\alpha)^3}{e^6 - (e^3+e+e^2)(tan\alpha)^2}$$

$$-e^6 tan\alpha + (e^3+e+e^2)(tan\alpha)^3 - e^3(1+e+e^2)tan\alpha + (tan\alpha)^3 = 0$$

$$-e^6 + (e^3+e+e^2)(tan\alpha)^2 - e^3(1+e+e^2) + (tan\alpha)^2 = 0$$

$$-e^6 + (1+e^3+e+e^2)(tan\alpha)^2 - e^3(1+e+e^2) = 0$$

$$(tan\alpha)^2 = \frac{e^3(1+e+e^2+e^3)}{(1+e+e^2+e^3)} = e^3$$

Therefore, $tan\alpha = e^{3/2}$ (21)

Hence the prove

To get the angle between DA and AB:

$$\tan(\alpha+\delta) = \frac{tan\alpha + tan\delta}{1 - tan\alpha tan\delta} \quad (22)$$

Substitute by (17) into (22)

$$\tan(\alpha+\delta)=\frac{tan\alpha+\frac{tan\alpha}{e^{3}}}{1-tan\alpha\frac{tan\alpha}{e^{3}}} \qquad (23)$$

Substitute by (21) into (23)

$$\tan(\alpha+\delta)=\frac{e^{3/2}+\frac{e^{3/2}}{e^{3}}}{1-\frac{e^{3}}{e^{3}}}=\frac{e^{3/2}+\frac{1}{e^{3/2}}}{1-1} \qquad (24)$$

It is clear that $\tan(\alpha+\delta)=\infty$, which means that $(\alpha+\delta)=\frac{\pi}{2}$

Hence the prove